Direction of Arrival Estimation with Hybrid Digital Analog Arrays: Algorithms and Ziv-Zakai Lower Bound

Miguel Rivas-Costa and Carlos Mosquera
atlanTTic Research Center, Universidade de Vigo, Vigo, Spain

***Abstract*—In this paper, the Ziv-Zakai Bound (ZZB) is applied to analyze the Direction of Arrival (DoA) estimation problem with hybrid arrays. Two signal models are considered: in the first one, the signal is constant and known, while in the second one, the signal varies over time and follows a Complex Gaussian distribution. The bounds are used as benchmarks for the MAP estimator, and through these bounds, the effect of the number of RF chains as well as the number of snapshots on the estimation problem is analyzed. Simulations demonstrate the bounds predict the Mean Squared Error (MSE) of MAP through the SNR.**

***Index Terms*—Direction of arrival estimation, array signal processing, maximum a posteriori estimation.**

I. Introduction

ESTIMATING the Direction of Arrival (DoA) is a major problem in array signal processing with importance in many field including radar, sonar or medical imaging [1]. Employing lower bounds in the minimum square error (MSE) is a usual approach to assessing the performance of the estimators. The so-called "local" bounds are widely used, and among them the most popular is the Cramer-Rao Bound (CRB), with handy expressions in some relevant problems, and which provides useful insights about the estimation performance. However, in the case of DoA estimation, a non-linear problem, the CRB is a useful benchmark only in the asymptotic region, i.e., under high SNR and/or high observation time [2]. To overcome this drawback, we can resort to the family of Bayesian bounds which provides a tighter alternative to the CRB. Some members of this family are the Bayesian Cramer Rao Bound (BCRB) [3, p. 84], the Weiss-Weinstein bound (WWB) [4], the Bayesian Bhattacharyya Bound (BBB) [5] or the Ziv-Zakai Bound (ZZB) [2]. This family assumes the parameters are random variables with known *a priori* distribution and the MSE is averaged over the distribution. A good compilation of results about local and non-local bounds can be found in [6]. In this work, our focus will be on the ZZB, which has been successfully applied to DoA estimation in [2], [7], and more recently in [8], extending the bound to multisource DoA estimation.

In the case of fully digital (FD) arrays, the estimation problem has been widely studied [9], [10]. Nevertheless, with the advent of massive multiple-input multiple-output (MIMO) systems, hybrid analog-digital arrays (HAD) employing fewer RF-chains than antennas have gained attention in recent years to reduce the hardware complexity and power consumption of FD arrays [11], [12]. Despite this, the study of lower bounds for DoA estimation with HAD arrays is not fully developed yet, in particular in the case of ZZB. In [1, p. 1073], we can find the expression of the CRB for Gaussian signals, referred to as "beamspace CRB" therein. In [13], the authors derived a closed form expression of CRB for impinging Gaussian signals using sub-connected HAD arrays and, in [14], an in-depth analysis of the CRB for deterministic signals was presented. Regarding Bayesian bounds, [8] provided an extension of the ZZB for FD arrays in multisource scenarios.

This work addresses the extension of the ZZB to DoA with HAD arrays. The role of the beamforming matrix, number of RF-chains, and the probability density function (pdf) of the angle of arrival will be discussed, along with the derivation of the Maximum a Posteriori Estimator (MAP) for HAD architectures.

II. System model

We assume an HAD array with N elements and N_{RF} RF-chains. A set of K, possibly different, beamforming matrices $\{\boldsymbol{V}_t\}$ is used across K different time snapshots. An incoming signal from angle θ yields the following sets of samples:

$$\boldsymbol{y}_t = \boldsymbol{V}_t^H(\boldsymbol{a}(u)s_t + \boldsymbol{n}_t) \quad t = 1, \dots, K \tag{1}$$

where $u \triangleq \sin\theta$ is the random directional sine with pdf $p(u)$, and $\boldsymbol{a}(u)$ is the array manifold or steering vector defined as:

$$\boldsymbol{a}(u) = \begin{pmatrix} 1 & \dots & e^{j2\pi\Delta(N-1)u} \end{pmatrix}^T. \tag{2}$$

The beamforming matrix $\boldsymbol{V}_t \in \mathbb{C}^{N\times N_{\text{RF}}}$ in (1) is assumed to be semiunitary $\boldsymbol{V}_t^H\boldsymbol{V}_t = \boldsymbol{I}_{N_{\text{RF}}}$. the noise $\boldsymbol{n}_t$ is white complex Gaussian with zero mean and covariance $\sigma_n^2\boldsymbol{I}_N$, and the signal s_t adopts two different formats: (i) constant and known $s_t = \sqrt{\sigma_s^2}$, (ii) complex random following a Gaussian distribution $s_t \sim \mathcal{CN}(0, \sigma_s^2)$. The signal to noise ratio is defined as $\mathsf{snr} \triangleq \frac{\sigma_s^2}{\sigma_n^2}$.

If we put together all the received samples as $\boldsymbol{Y} \triangleq (\boldsymbol{y}_1, \dots, \boldsymbol{y}_K)^T$, then the ZZB expression used in this paper reads as [2]:

$$\begin{aligned} &\mathrm{E}_{\boldsymbol{Y},u}\left[(\hat{u}(\boldsymbol{Y}) - u)^2\right] \\ &\geq \int_0^\infty \mathcal{V}\left\{\int_{-\infty}^\infty \min(p(u), p(u+h))P_{\min}(u, u+h)du\right\} hdh \end{aligned} \tag{3}$$

where $\mathcal{V}\{\cdot\}$ is the valley-filling function and $P_{\min}(u, u+h)$ is the minimum probability of error in the binary detection problem of deciding between u and $u+h$. If $P_{\text{min}}(u, u+h)$ depends only on h, Eq. (3) admits the simplification

$$\mathrm{E}_{\boldsymbol{Y},u}\left[(\hat{u}(\boldsymbol{Y})-u)^2\right] \geq \int_0^\infty \mathcal{V}\{A(h)P_{\text{min}}(h)\}hdh \quad (4)$$

$$A(h) = \int_{-\infty}^{\infty} \min(p(u), p(u+h))du. \quad (5)$$

III. ZZB FOR CONSTANT SIGNALS

For known $s_t = \sqrt{\sigma_s^2}\ \ \forall t$, the expectation and covariance of $\boldsymbol{y}_t$ are $\boldsymbol{\mu}_t(u) = \sqrt{\sigma_s^2}\boldsymbol{V}_t^H\boldsymbol{a}(u)$ and $\boldsymbol{\Sigma}_t(u) = \sigma_n^2\boldsymbol{I}_{N_{\text{RF}}}$, respectively, so the pdf of $\boldsymbol{Y}$ given u boils down to

$$p(\boldsymbol{Y}|u) = (\pi\sigma_n^2)^{-KN_{\text{RF}}}e^{-\frac{1}{\sigma_n^2}\sum_{t=1}^K ||\boldsymbol{y}_t - \sqrt{\sigma_s^2}\boldsymbol{V}_t^H\boldsymbol{a}(u)||_2^2}. \quad (6)$$

The minimum probability of error is attained with the Log Likelihood Ratio Test (LLRT)

$$\ell(\boldsymbol{Y}) = \ln p(\boldsymbol{Y}|u+h) - \ln p(\boldsymbol{Y}|u) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} 0 \quad (7)$$

which, after substituting (6) in (7), results in the statistic

$$\ell(\boldsymbol{Y}) = \sum_{t=1}^K \text{Re}\left[(\boldsymbol{\mu}_t(u+h) - \boldsymbol{\mu}_t(u))^H\boldsymbol{y}_t\right] \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \gamma \quad (8)$$

where $\gamma = \frac{1}{2}\sum_{t=1}^K\left\{||\boldsymbol{\mu}_t(u+h)||_2^2 - ||\boldsymbol{\mu}_t(u)||_2^2\right\}$. With this, the minimum probability of error reduces to

$$P_{\text{min}}(u,u+h) = 0.5\big[P(\ell(\boldsymbol{Y}|u) \geq \gamma) + P(\ell(\boldsymbol{Y}|u+h) < \gamma). \quad (9)$$

Given that $\ell(\boldsymbol{Y})$ in (8) is the real part of a linear transformation of a complex Gaussian vector, $\ell(\boldsymbol{Y}|u)$ and $\ell(\boldsymbol{Y}|u+h)$ will be real Gaussian random variables, so we can resort to the Gaussian $\mathcal{Q}(\cdot)$ function to obtain $P_{\text{min}}(u, u+h)$. Letting $\mu_{\ell|u} \triangleq \mathrm{E}\left[\ell(\boldsymbol{Y}|u)\right]$, the expectations of $\ell(\boldsymbol{Y}|u)$ and $\ell(\boldsymbol{Y}|u+h)$ are

$$\mu_{\ell|u} = \sum_{t=1}^K \text{Re}\left[(\boldsymbol{\mu}_t(u+h) - \boldsymbol{\mu}_t(u))^H\boldsymbol{\mu}_t(u)\right] \quad (10)$$

$$\mu_{\ell|u+h} = \sum_{t=1}^K \text{Re}\left[(\boldsymbol{\mu}_t(u+h) - \boldsymbol{\mu}_t(u))^H\boldsymbol{\mu}_t(u+h)\right] \quad (11)$$

with the same variance in both cases:

$$\sigma_\ell^2 = \frac{\sigma_n^2}{2}\sum_{t=1}^K ||\boldsymbol{\mu}_t(u+h) - \boldsymbol{\mu}_t(u)||_2^2. \quad (12)$$

Finally, using (10)-(12), Eq. (9) reads $P_{\text{min}}(u, u+h) =$

$$0.5\Big[\mathcal{Q}\left(\frac{\gamma - \mu_{\ell|u}}{\sigma_l}\right) + 1 - \mathcal{Q}\left(\frac{\gamma - \mu_{\ell|u+h}}{\sigma_l}\right)\Big]. \quad (13)$$

In order to apply the simplified ZZB expression (4), we identify the beamforming matrices such that the Fisher Information Matrix (FIM) is independent on u. From (47) in Appendix-A, a sufficient condition is the following **identity resolution**:

$$\sum_{t=1}^K \boldsymbol{V}_t\boldsymbol{V}_t^H = \frac{KN_{\text{RF}}}{N}\boldsymbol{I}_N. \quad (14)$$

Note that the constant $\frac{N_{\text{RF}}}{N}$ follows from the semiunitary property of $\boldsymbol{V}_t$. With this, it can be readily seen that

$$\sum_{t=1}^K ||\boldsymbol{\mu}_t(u)||_2^2 = \sigma_s^2\sum_{t=1}^K \boldsymbol{a}^H(u)\boldsymbol{V}_t\boldsymbol{V}_t^H\boldsymbol{a}(u) = K\sigma_s^2N_{\text{RF}} \quad (15)$$

$$\sum_{t=1}^K \boldsymbol{\mu}_t^H(u)\boldsymbol{\mu}_t(u+h) = K\sigma_s^2N_{\text{RF}}G(h) \quad (16)$$

where $G(h) \triangleq \frac{1}{N}\boldsymbol{a}^H(u)\boldsymbol{a}(u+h)$. If (15) is applied in the LLRT (8), the threshold reduces to $\gamma = 0$. Besides, Eqs. (10) and (11) become

$$\mu_{\ell|u} = -\mu_{\ell|u+h} = KN_{\text{RF}}\sigma_s^2(\text{Re}\left[G(h)\right] - 1) \quad (17)$$

and the variance reads

$$\sigma_l^2 = K\sigma_n^2\sigma_s^2N_{\text{RF}}(1 - \text{Re}\left[G(h)\right]). \quad (18)$$

Finally, substituting (10) and (11) in (13), the minimum probability of error is written as

$$P(h) = \mathcal{Q}\left(\sqrt{K \cdot N_{\text{RF}} \cdot \mathsf{snr} \cdot (1 - \text{Re}\left[G(h)\right])}\right). \quad (19)$$

IV. ZZB FOR GAUSSIAN SIGNALS

If $s_t \sim \mathcal{CN}(0, \sigma_s^2)$, the received signal $\boldsymbol{y}_t$ is characterized as a complex Gaussian distribution with zero mean and covariance

$$\boldsymbol{\Sigma}_t(u) = \sigma_s^2\boldsymbol{V}_t^H\boldsymbol{a}(u)\boldsymbol{a}(u)^H\boldsymbol{V}_t + \sigma_n^2\boldsymbol{I}_{N_{\text{RF}}}. \quad (20)$$

The pdf of $\boldsymbol{Y}$ given u is expressed as

$$p(\boldsymbol{Y}|u) = \pi^{-KN_{\text{RF}}}\prod_{t=1}^K |\boldsymbol{\Sigma}_t(u)|^{-1} e^{-\sum_{t=1}^K \boldsymbol{y}_t^H\boldsymbol{\Sigma}_t^{-1}(u)\boldsymbol{y}_t} \quad (21)$$

and the inverse covariance

$$\boldsymbol{\Sigma}_t^{-1}(u) = \frac{1}{\sigma_n^2}\left(\boldsymbol{I}_{N_{\text{RF}}} - \frac{\mathsf{snr}\cdot\boldsymbol{V}_t^H\boldsymbol{a}(u)\boldsymbol{a}(u)^H\boldsymbol{V}_t}{1 + ||\boldsymbol{V}_t^H\boldsymbol{a}(u)||_2^2\cdot\mathsf{snr}}\right). \quad (22)$$

In contrast to the constant model, the statistic of the LLRT for the Gaussian model follows a Generalized Chi-Square (GCS) distribution [3], where $P_{\text{min}}(u, u+h)$ does not admit, in general, a closed form expression. Hence, we will employ the lower bound developed in [3, p. 125]:

$$\begin{aligned} &P_{\text{min}}(u, u+h) \geq P(u, h) \quad (23)\\ &= \frac{1}{2}\exp\left\{\mu(x^*) + \frac{(x^*)^2}{2}\ddot{\mu}(x^*)\right\}\mathcal{Q}\left(x^*\sqrt{\ddot{\mu}(x^*)}\right)\\ &+ \frac{1}{2}\exp\left\{\mu(x^*) + \frac{(1-x^*)^2}{2}\ddot{\mu}(x^*)\right\}\mathcal{Q}\left((1-x^*)\sqrt{\ddot{\mu}(x^*)}\right) \end{aligned}$$

where $\mu(x, h)$ is the semi-invariant moment generating function defined as

$$\mu(x, h) = \ln\int_{-\infty}^{\infty} p(\boldsymbol{Y}|u+h)^x p(\boldsymbol{Y}|u)^{1-x}d\boldsymbol{Y}. \quad (24)$$

Substituting the pdf (21) into (24) reduces to [7]

$$\begin{aligned} \mu(x, h) = \sum_{t=1}^K \Big\{&x\ln|\boldsymbol{\Sigma}_t(u)| + (1-x)\ln|\boldsymbol{\Sigma}_t(u+h)|\\ &- \ln|x\boldsymbol{\Sigma}_t(u) - (1-x)\boldsymbol{\Sigma}_t(u+h)|\Big\}. \quad (25) \end{aligned}$$

Determinants are evaluated as [15]

$$|\boldsymbol{\Sigma}_t(u)| = (1+\mathsf{snr}\cdot xN_t^u)(\sigma_n^2)^{N_{\text{RF}}} \quad (26)$$

$$|x\boldsymbol{\Sigma}_t(u)+(1-x)\boldsymbol{\Sigma}_t(u+h)| = (\sigma_n^2)^{N_{\text{RF}}}\big[\big(1+\mathsf{snr}\cdot xN_t^u\big) \cdot \big(1+\mathsf{snr}\cdot xN_t^{u+h}\big) - x(1-x)\mathsf{snr}^2\cdot N_t^{uh}\big] \quad (27)$$

where we are using the following definitions:

$$N_t^u \triangleq ||\boldsymbol{V}_t^H\boldsymbol{a}(u)||_2^2 \quad (28)$$

$$N_t^{uh} \triangleq |\boldsymbol{a}^H(u)\boldsymbol{V}_t\boldsymbol{V}_t^H\boldsymbol{a}(u+h)|^2 \quad (29)$$

By means of (26) and (27), $\mu(x,h)$ reduces to

$$\mu(x,h) = \sum_{t=1}^{K}\Big\{x\ln\Big(1+\mathsf{snr}\cdot N_t^u\Big)+(1-x)\ln\Big(1+\mathsf{snr}\cdot N_t^{u+h}\Big) - \ln\Big(\Big(1+\mathsf{snr}\cdot xN_t^u\Big)\Big(1+\mathsf{snr}\cdot(1-x)N_t^{u+h}\Big)-\mathsf{snr}^2\cdot x(1-x)N_t^{uh}\Big)\Big\} \quad (30)$$

with first derivative equal to

$$\dot{\mu}(x) = \sum_{t=1}^{K}\Bigg\{\ln\Big(1+\mathsf{snr}\cdot N_t^u\Big)-\ln\Big(1+\mathsf{snr}\cdot N_t^{u+h}\Big) + \frac{\mathsf{snr}^2\cdot(1-2x)(N_t^{uh}-N_t^uN_t^{u+h})+\mathsf{snr}\cdot(N_t^{u+h}-N_t^u)}{(1+\mathsf{snr}\cdot xN_t^u)(1+\mathsf{snr}\cdot(1-x)N_t^{u+h})-\mathsf{snr}^2\cdot x(1-x)N_t^{uh}}\Bigg\} \quad (31)$$

and the second one reads

$$\ddot{\mu}(x) = \sum_{t=1}^{K}\Bigg\{\frac{\mathsf{snr}^2\cdot 2(N_t^uN_t^{u+h}-N_t^{uh})}{(1+\mathsf{snr}\cdot xN_t^u)(1+\mathsf{snr}\cdot(1-x)N_t^{u+h})-\mathsf{snr}^2\cdot x(1-x)N_t^{uh}} + \frac{(\mathsf{snr}^2\cdot(1-2x)(N_t^{uh}-N_t^uN_t^{u+h})+\mathsf{snr}\cdot(N_t^{u+h}-N_t^u))^2}{((1+\mathsf{snr}\cdot xN_t^u)(1+\mathsf{snr}\cdot(1-x)N_t^{u+h})-\mathsf{snr}^2\cdot x(1-x)N_t^{uh})^2}\Bigg\}. \quad (32)$$

As opposed to the constant signal case described in the previous section, FIM independence of u cannot be guaranteed. In this regard, from the expression of FIM derived in Appendix -B, a sufficient condition would be

$$\boldsymbol{V}_t\boldsymbol{V}_t^H = \frac{N_{\text{RF}}}{N}\boldsymbol{I}_N \quad (33)$$

which cannot be enforced unless $N_{RF}=N$. Therefore, unlike the constant signal case, the ZZB for the signal must be calculated without using the simplification (4).

V. Description of performance regions

The performance of DoA estimators can be described according to three different regions: low-SNR, high-SNR, and a transition region in between. For low SNR, the error is dominated by the *a priori* distribution of u, whereas for high SNR the error converges to local bounds like CRB. In [7], the authors noted that if the CRB is independent of u, the ZZB converges to the CRB, and used it to locate the boundary points: from low-SNR to transition and from transition to high-SNR. For space reasons, we do not delve further into this, and refer the interested readers to [7] for additional details.

For notation convenience we introduce the definition $\overline{A} \triangleq \int_0^\infty A(h)h\,dh$, and h_z such that $G(h_z)=0$. For low SNR, the ZZB is upper bounded by $0.5\overline{A}$ [7], which only depends on $p(u)$. The evaluation of $\overline{A}$ for uniformly distributed u was presented in [7], and it is extended here in Appendix-C for uniform θ. Following [7] for the constant signal model, we seek the SNR where the ZZB gets 3dB below the upper bound in low SNR, $P_{\text{min}}(h_z)=0.5$, obtained as the rightmost solution of $\overline{A}\cdot P_{\text{min}}(h_z)=J^{-1}$. For high SNR, the FIM for the constant model reads

$$J_{\text{C}} = 2K(2\pi\Delta)^2\mathsf{snr}\cdot N_{\text{RF}}\frac{S_{n^2}}{N} \quad (34)$$

where $S_{n^2} = \sum_{n=0}^{N-1}n^2$. The impact of the number of snapshots can be illustrated by comparing the bounds for two architectures with K_1 and K_2 snapshots. For the constant signal case, we have

$$\frac{\text{CRB}_{\text{C},1}}{\text{CRB}_{\text{C},2}} = \frac{J_{C,2}}{J_{C,1}} = \frac{K_2}{K_1}. \quad (35)$$

As stated in Section IV, the ZZB in the Gaussian case can not be simplified, therefore the transition points between regions are hard to identify. To obtain some insights, we focus our interest on the case where the beamforming matrix is random $[\boldsymbol{V}]_{n,m} = \frac{1}{\sqrt{N}}e^{j\phi_{n,m}}$ with $\phi_{n,m}\sim\mathcal{U}[0,2\pi)$. At high SNR, the CRB (51) can be approximated, for high N_{RF} and K, as

$$J_G \approx 2K(2\pi\Delta)^2\mathsf{snr}\Big(\frac{N_{\text{RF}}}{N}S_n^2 - \frac{\text{E}\left[|\boldsymbol{a}\boldsymbol{D}\boldsymbol{V}_t\boldsymbol{V}_t^H\boldsymbol{a}|^2\right]}{\text{E}\left[||\boldsymbol{V}_t^H\boldsymbol{a}||_2^2\right]}\Big). \quad (36)$$

when N_{RF} is high and K increases. Once the expectations are computed, the FIM reads as

$$J_G \approx 2K(2\pi\Delta)^2\mathsf{snr}\frac{N_{\text{RF}}}{N}\Big(\frac{N-1}{N}S_{n^2} - \frac{S_n^2}{N}\Big). \quad (37)$$

VI. Maximum a Posteriori (MAP)

In this work we use the ZZB for benchmarking purposes of the maximum a posteriori (MAP) estimator $\hat{u}_{\text{MAP}}$:

$$\hat{u}_{\text{MAP}} = \arg\max_u\, p(u|\boldsymbol{Y}) = \arg\max_u\, p(\boldsymbol{Y}|u)p(u), \quad (38)$$

or, for convenience, expressed as

$$\hat{u}_{\text{MAP}} = \arg\max_u[\ln p(\boldsymbol{Y}|u)+\ln p(u)]. \quad (39)$$

The corresponding MAP estimators for the constant and Gaussian models will be labeled as $\hat{u}_{\text{C}}$ and $\hat{u}_{\text{G}}$, respectively[1]. Starting with the constant model, the pdf (6) is substituted in (39) and MAP reduces to

$$\hat{u}_{\text{C}} = \arg\max_u\big[-KN_{\text{RF}}\ln\pi - \sum_{t=1}^{K}||\boldsymbol{y}_t-\sqrt{\sigma_s^2}\boldsymbol{V}_t^H\boldsymbol{a}(u)||_2^2+\ln p(u)\big], \quad (40)$$

or, alternatively,

$$\hat{u}_{\text{C}} = \arg\max_u\Bigg[\sum_{t=1}^{K}\Big\{2\sqrt{\sigma_s^2}\text{Re}\left[\boldsymbol{y}_t^H\boldsymbol{V}_t^H\boldsymbol{a}(u)\right] - \sigma_s^2\boldsymbol{a}(u)\boldsymbol{V}_t\boldsymbol{V}_t^H\boldsymbol{a}(u)\Big\}+\ln p(u)\Bigg]. \quad (41)$$

[1]Note that for a priori uniformly distributed u, MAP coincides with the maximum likelihood estimator.

Similarly for the Gaussian signal model, after substituting the pdf (21) in (39), the estimator is given by maximising

$$-KN_{\text{RF}}\ln\pi - \sum_{t=1}^{K}\left\{\ln|\boldsymbol{\Sigma}_t(u)| + \boldsymbol{y}_t^H\boldsymbol{\Sigma}_t^{-1}(u)\boldsymbol{y}_t\right\} + \ln p(u)$$

which can be conveniently simplified to yield

$$\hat{u}_{\text{G}} = \arg\max_{u}\left[\sum_{t=1}^{K}\left\{\frac{||\boldsymbol{y}_t^H\boldsymbol{V}_t^H\boldsymbol{a}(u)||_2^2}{\sigma_n^2(1+\mathsf{snr}N_t^u)} - \ln(1+\mathsf{snr}N_t^u)\right\} + \ln p(u)\right]. \qquad (42)$$

As a remark, note that this work applies to DoA estimation based on independent snapshots. For those schemes which make use of Bayesian Learning [16], [17], and choose $\boldsymbol{V}_t$ adaptively as a function of the observations, bounds such as CRB or ZZB do not apply.

VII. Simulations

Fig. 1a depicts the ZZB curves along with performance of MAP for $5\cdot10^3$ realizations, obtained for the constant signal model using DFT beamforming, $[\boldsymbol{V}_t]_{n,m} = \frac{1}{\sqrt{N}}e^{j2\pi\Delta n u_m}$ $n=0,\dots,N-1$, where u_m denotes the center of the m-th beamforming vector, spaced by steps $\frac{2}{N}$. Since the beamforming satisfies the identity resolution condition, we know that the CRB bound is independent of the direction, and therefore, ZZB converges to CRB at high SNR. At low SNR, the bound is dominated by the effect of the prior distribution of u, with a threshold of $\frac{1}{3}$ when u is uniform and $\frac{1}{4}$ when θ is uniform. Fig. 1a showcases different number of snapshots, $K = \{20, 40, 80\}$, which, according to our theoretical derivations, results in a constant separation between bounds by a factor of two at high SNR.

Fig. 1b shows the simulation of ZZB and MAP with $5\cdot10^3$ realizations for the Gaussian model using random beamforming $[\boldsymbol{V}]_{n,m} = \frac{1}{\sqrt{N}}e^{j\phi_{n,m}}$, with $\phi\sim\mathcal{U}[0,2\pi)$. The straight line is the approximation for the CRB in high SNR, whereas the horizontal lines correspond to the thresholds for low SNR. Similarly to the constant signal case, at low SNR, the bound is dominated by the a priori distribution of u; as the SNR increases, the bound tends towards the CRB because random beamforming asymptotically makes the CRB independent of the DoA. Finally, it is worth noting that, for high SNR, there is an inverse dependence between the number of RF chains N_{RF} and the MSE, as anticipated by expression (37).

VIII. Conclusions

The Ziv-Zakai Bound (ZZB) for DoA with hybrid arrays has been developed for the constant and known signal model. It has been shown that the satisfaction of the so-called identity resolution condition by the beamforming matrices allows for a simplified evaluation of the bound, including the identification of the transition between regions and its asymptotic expression for high SNR. For the Gaussian signal model, if random beamforming is applied, a convenient high SNR expression has been derived, which allows to understand the role of the number of snapshots, antennas and RF chains.

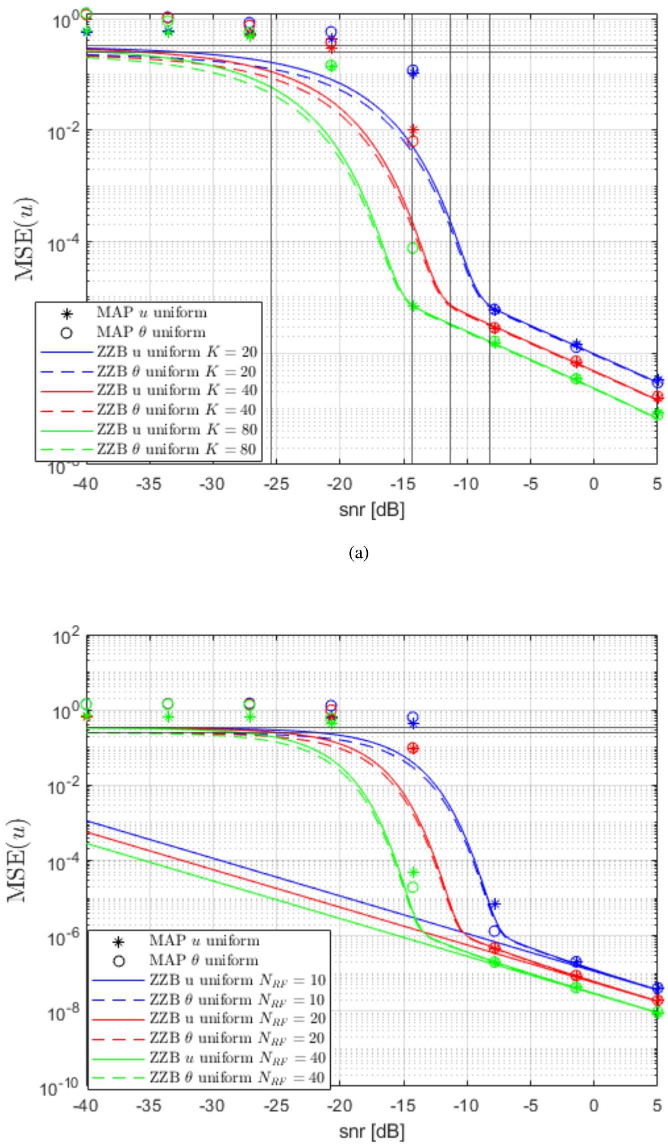

Fig. 1. (a) Constant signal model. $N = 32$, $N_{\text{RF}} = 8$, $\Delta = \frac{1}{2}$ (b) Gaussian signal model. $N = 120$, $\Delta = \frac{7}{16}$, $K = 50$.

Appendix

A. Derivation of FIM for the constant signal model

For complex Gaussian data with mean $\boldsymbol{\mu}_{\boldsymbol{Y}}(u)$ and covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{Y}}$ independent of u, the FIM is given by [1]:

$$J(u) = 2\text{Re}\left[\frac{\partial\boldsymbol{\mu}_{\boldsymbol{Y}}^H(u)}{\partial u}\boldsymbol{\Sigma}_{\boldsymbol{Y}}^{-1}(u)\frac{\partial\boldsymbol{\mu}_{\boldsymbol{Y}}(u)}{\partial u}\right]. \qquad (43)$$

In the constant signal model the mean of the data vector is $\boldsymbol{\mu}_{\boldsymbol{Y}}(u) = (s\boldsymbol{V}_1^H\boldsymbol{a}(u),\dots,s\boldsymbol{V}_K^H\boldsymbol{a}(u))^T$ and the covariance is $\boldsymbol{\Sigma}_{\boldsymbol{Y}} = \sigma_n^2\boldsymbol{I}_{KN_{\text{RF}}}$, so the FIM, denoted by J_{C}, reduces to

$$J_{\text{C}}(u) = \frac{2}{\sigma_n^2}\sum_{t=1}^{K}\text{Re}\left[\frac{\partial\boldsymbol{\mu}_t^H(u)}{\partial u}\frac{\partial\boldsymbol{\mu}_t(u)}{\partial u}\right]. \qquad (44)$$

Let the derivative of the steering vector with respect to u be

$$\dot{\boldsymbol{a}}(u) \triangleq \frac{\partial \boldsymbol{a}(u)}{\partial u} = j2\pi\Delta \boldsymbol{D}\boldsymbol{a}(u) \tag{45}$$

with $\boldsymbol{D} = \text{diag}(0, \ldots, N-1)$, from which the derivatives in (44) follow:

$$\frac{\partial \boldsymbol{\mu}_t(u)}{\partial u} = j\sqrt{\sigma_s^2} 2\pi\Delta \boldsymbol{V}_t^H \boldsymbol{D}\boldsymbol{a}(u) \tag{46}$$

and the FIM reads as

$$J_{\text{C}}(u) = 2(2\pi\Delta)^2 \mathsf{snr} \cdot \boldsymbol{a}^H(u)\boldsymbol{D}\sum_{t=1}^{K} \boldsymbol{V}_t \boldsymbol{V}_t^H \boldsymbol{D}\boldsymbol{a}(u). \tag{47}$$

B. Derivation of FIM for the Gaussian signal model

For the Gaussian model presented in Section IV the vector of data $\boldsymbol{Y}$ is Gaussian with zero mean and the covariance matrix is block diagonal where each block is given by (20). Therefore, the FIM (43), denoted as $J_{\text{G}}(u)$, reduces to

$$J_{\text{G}}(u) = \sum_{t=1}^{K} \text{trace}\left[\boldsymbol{\Sigma}_t^{-1}(u)\frac{\partial \boldsymbol{\Sigma}_t(u)}{\partial u}\boldsymbol{\Sigma}_t^{-1}(u)\frac{\partial \boldsymbol{\Sigma}_t(u)}{\partial u}\right]. \tag{48}$$

The derivative of the covariance matrix with respect to u is

$$\frac{\partial \boldsymbol{\Sigma}_t(u)}{\partial u} = \sigma_s^2 \boldsymbol{V}_t^H(\dot{\boldsymbol{a}}^H(u)\boldsymbol{a}(u) + \boldsymbol{a}^H(u)\dot{\boldsymbol{a}}(u))\boldsymbol{V}_t. \tag{49}$$

For the sake of notation, we will drop the term u in the following equations. By plugging (49) into (48), J_{G} simplifies to

$$J_{\text{G}} = 2(2\pi\Delta)^2 \sum_{t=1}^{K}\left\{\frac{\mathsf{snr}^2 N_t^u}{1+\mathsf{snr}N_t^u}||\boldsymbol{V}_t^H\boldsymbol{D}\boldsymbol{a}||_2^2 - \frac{\mathsf{snr}^2\text{Re}\left[\boldsymbol{a}^H\boldsymbol{D}\boldsymbol{V}_t\boldsymbol{V}_t^H\boldsymbol{a}\right]^2}{1+\mathsf{snr}N_t^u}\right.$$
$$\left. - \frac{\mathsf{snr}^2(\mathsf{snr}N_t^u - 1)\text{Im}\left[\boldsymbol{a}^H\boldsymbol{D}\boldsymbol{V}_t\boldsymbol{V}_t^H\boldsymbol{a}\right]^2}{(1+\mathsf{snr}N_t^u)^2}\right\} \tag{50}$$

which, for high SNR, can be approximated by

$$J_G = 2(2\pi\Delta)^2 \mathsf{snr}\sum_{t=1}^{K}\left\{||\boldsymbol{V}_t^H\boldsymbol{D}\boldsymbol{a}||_2^2 - \frac{|\boldsymbol{a}\boldsymbol{D}\boldsymbol{V}_t\boldsymbol{V}_t^H\boldsymbol{a}|^2}{||\boldsymbol{V}_t^H\boldsymbol{a}||_2^2}\right\}. \tag{51}$$

It should be noted that the first term in (51) is the FIM of the constant case.

C. Evaluation of $\overline{A}(h)$

In this Appendix we present the evaluation of $\overline{A} \triangleq \int_0^\infty A(h)hdh = \int_0^\infty \int_{-\infty}^\infty \min(p(u), p(u+h))duhdh$ when θ is uniformly distributed as $\theta \sim \mathcal{U}(-\theta_c, \theta_c)$ with $\theta_c > 0$, and u follows an arcsin distribution:

$$p(u) = \frac{1}{2\theta_c}\frac{1}{\sqrt{1-u^2}} \quad u \in (-u_c, u_c). \tag{52}$$

with $u_c = \sin\theta_c$. As the function $\min(p(u), p(u+h))$ is piecewise and can be splitted as

$$\min\left[p(u), p(u+h)\right] = \begin{cases} p(u+h) & -u_c < u \leq -\frac{h}{2} \\ p(u) & -\frac{h}{2} < u < u_c - h \\ 0 & \text{other} \end{cases}$$

then $A(h)$ reads

$$\begin{aligned} A(h) &= \int_{-\infty}^{\infty} \min\left[p(u), p(u+h)\right] du \\ &= \frac{2}{\pi}(\arcsin(\tfrac{h}{2}) - \arcsin(h-u_c)) \quad h \in [0, 2u_c]. \end{aligned} \tag{53}$$

Now, the integral $\int_0^{2u_c} A(h)hdh$ can be readily obtained. For example, for $\theta_c = \frac{\pi}{2}$, we have

$$\overline{A} = \int_0^2 A(h)hdh = \frac{1}{2}. \tag{54}$$

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