

# A CLASS OF CIRCULAR SPARSE RULERS FOR COMPRESSIVE POWER SPECTRUM ESTIMATION

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## ABSTRACT

Power Spectrum Blind Sampling (PSBS) has been recently proposed as a strategy to reduce sampling rates below the Nyquist frequency when the objective is not perfect signal reconstruction but estimation of the power spectrum. In addition, sparse rulers have been proposed as multicoset patterns that can be used in PSBS to reconstruct the power spectrum even when the signal is not sparse. In this paper we propose a sparse ruler design method based on coprime sampling, which gives interesting solutions when long patterns have to be designed.

**Index Terms**— Compressive sampling, spectrum sensing, power spectrum blind sampling, multicoset sampling, minimal sparse ruler

## 1. INTRODUCTION

Compressive Covariance Sampling or equivalently Power Spectrum Blind Sampling is an interesting strategy to reconstruct the power spectrum of a given signal from sub-Nyquist rate samples [1,2]. Its application is clear in the area of spectrum sensing for cognitive radio [3], since we are dealing with wideband signals and traditional sampling devices at the corresponding Nyquist rate can not be implemented.

Although compressive sampling strategies [4,5] are closely related to PSBS, it is not necessary to assume sparsity of the spectrum in order to apply a PSBS strategy, as it has already been shown [1,2]. Compressed measurements can be obtained in different ways to reconstruct the power spectrum, although in this paper we will focus on *multicoset sampling* techniques [6,7].

Ariananda and Leus [1,2] have previously obtained a sufficient condition for a multicoset sampling pattern can be used to reconstruct the power spectrum with a low complexity solution. In the same works they have proposed *minimal sparse*

*rulers* [8] as a solution for the multicoset design problem. In this context, Romero and Leus [9] have developed very recently a new condition for perfect spectrum reconstruction, proposing what they call a *circular sparse ruler*.

In this paper, we show that the conditions of Ariananda and Leus can be weakened, and we obtain an alternative design method for the sparse ruler which is based on coprime sampling as in [10,11]. This new design can be classified as a particular family of circular sparse rulers as defined in [9]. The main advantage of the new design proposed here is that long patterns can be easily designed and used for Power Spectrum Blind Sampling.

## 2. SAMPLING MODEL AND PROBLEM STATEMENT

In this paper we consider a multicoset sampling strategy [6,7] implemented with coprime sampling [10,11]. The input to the sampling stage is a complex-valued wide-sense stationary signal  $x(t)$  with bandwidth  $B$ . The final goal of the scheme is to use the multicoset samples obtained by coprime sampling to estimate the power spectrum of  $x(t)$ .

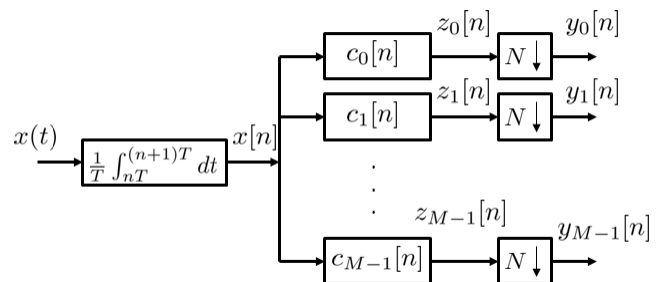


Fig. 1. Digital model of the sampling device.

This sampling device can be modeled as in [2]: a high rate integrate and dump process followed by a bank of  $M$  branches, each one consisting of a filtering operation followed by a downsampling operation, as illustrated in Figure 1. Taking into account that multicoset sampling consists of selecting

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a Nyquist-rate sample  $n_i$  in each block  $i$ , the coefficients of the filter  $c_i[n]$  can be written as

$$c_i[n] = \begin{cases} 1, & n = -n_i, \\ 0, & n \neq -n_i, \end{cases} \quad (1)$$

where there is no repetition in  $n_i$ , i.e.  $n_i \neq n_j, \forall i \neq j$ .

The output of the  $i$ th branch of this PSBS scheme is given by

$$y_i[k] = z_i[kN], \quad (2)$$

where  $z_i[\cdot]$  is given by

$$\begin{aligned} z_i[n] &= c_i[n] * x[n] \\ &= \sum_{m=1-N}^0 c_i[m]x[n-m]. \end{aligned} \quad (3)$$

It is shown in [2] that

$$\mathbf{r}_y = \mathbf{R}_c \mathbf{r}_x, \quad (4)$$

where  $\mathbf{r}_y \in \mathbb{C}^{\frac{1}{2}M(2L+1)(M+1) \times 1}$  and  $\mathbf{r}_x \in \mathbb{C}^{N(2L+1) \times 1}$  are given by

$$\mathbf{r}_y = [\mathbf{r}_y^T[0] \cdots \mathbf{r}_y^T[L] \mathbf{r}_y^T[-L] \cdots \mathbf{r}_y^T[-1]]^T, \quad (5a)$$

$$\mathbf{r}_x = [\mathbf{r}_x^T[0] \cdots \mathbf{r}_x^T[L] \mathbf{r}_x^T[-L] \cdots \mathbf{r}_x^T[-1]]^T, \quad (5b)$$

where  $L$  is a design parameter related to the support of  $\mathbf{r}_x[k]$  and  $\mathbf{R}_c \in \mathbb{C}^{\frac{1}{2}M(2L+1)(M+1) \times N(2L+1)}$  is given by

$$\mathbf{R}_c = \begin{bmatrix} \mathbf{R}_c[0] & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{R}_c[1] \\ \mathbf{R}_c[1] & \mathbf{R}_c[0] & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_c[1] & \mathbf{R}_c[0] & \mathbf{O} & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{R}_c[1] & \mathbf{R}_c[0] \end{bmatrix}. \quad (6)$$

The quantities  $\mathbf{r}_y[\cdot] \in \mathbb{C}^{\frac{1}{2}M(M+1) \times 1}$ ,  $\mathbf{r}_x[\cdot] \in \mathbb{C}^{N \times 1}$ , and  $\mathbf{R}_c[\cdot] \in \mathbb{C}^{\frac{1}{2}M(M+1) \times N}$  are related to each other by

$$\mathbf{r}_y[k] = \sum_{l=0}^1 \mathbf{R}_c[l] \mathbf{r}_x[k-l], \quad (7)$$

where

$$\mathbf{r}_y[k] = [r_{y_1, y_1}[k] \cdots r_{y_1, y_M}[k] r_{y_2, y_2}[k] \cdots r_{y_M, y_M}[k]]^T, \quad (8a)$$

$$\mathbf{R}_c[k] = [\mathbf{r}_{c_1, c_1}[k] \cdots \mathbf{r}_{c_1, c_M}[k] \mathbf{r}_{c_2, c_2}[k] \cdots \mathbf{r}_{c_M, c_M}[k]]^T, \quad (8b)$$

$$\mathbf{r}_x[k] = [r_x[kN] r_x[kN+1] \cdots r_x[(k+1)N-1]]^T. \quad (8c)$$

The quantities  $r_{y_i, y_j}[\cdot] \in \mathbb{C}^{1 \times 1}$  and  $r_x[\cdot] \in \mathbb{C}^{1 \times 1}$  are given by

$$r_{y_i, y_j}[k] = \mathcal{E}_y \{y_i[l]y_j^*[l-k]\}, \quad (9a)$$

$$r_x[n] = \mathcal{E}_x \{x[m]x^*[m-n]\}, \quad (9b)$$

and  $\mathbf{r}_{c_i, c_j}[\cdot] \in \mathbb{C}^{N \times 1}$  can be written as

$$\begin{aligned} \mathbf{r}_{c_i, c_j}[k] &= [r_{c_i, c_j}[kN] \ r_{c_i, c_j}[kN-1] \ \cdots \\ &\quad r_{c_i, c_j}[(k-1)N+1]]^T, \end{aligned} \quad (10)$$

where  $r_{c_i, c_j}[n] \in \mathbb{C}^{1 \times 1}$  is given by

$$r_{c_i, c_j}[n] = \sum_{m=1-N}^0 c_i[m]c_j^*[m-n]. \quad (11)$$

The number of cross-correlation functions to be considered for the PSBS is changed from  $\frac{1}{2}M(M+1)$  to  $M^2$  in [2]. This extension increases the sizes of  $\mathbf{r}_y[\cdot] \in \mathbb{C}^{M^2 \times 1}$  and  $\mathbf{R}_c[\cdot] \in \mathbb{C}^{M^2 \times N}$  according to

$$\mathbf{r}_y[k] = [r_{y_1, y_1}[k] \cdots r_{y_1, y_M}[k] r_{y_2, y_1}[k] \cdots r_{y_M, y_M}[k]]^T, \quad (12a)$$

$$\mathbf{R}_c[k] = [\mathbf{r}_{c_1, c_1}[k] \cdots \mathbf{r}_{c_1, c_M}[k] \mathbf{r}_{c_2, c_1}[k] \cdots \mathbf{r}_{c_M, c_M}[k]]^T. \quad (12b)$$

The condition for perfect power spectrum reconstruction obtained by Ariananda and Leus, is that this matrix  $\mathbf{R}_c$  has full column rank. When the sampling pattern is given by the set  $S$  of  $M$  samples

$$0 \leq n_0 < n_1 < \cdots < n_{M-1} \leq N-1,$$

substituting (1) into (11), we obtain<sup>1</sup>

$$\begin{aligned} r_{c_i, c_j}[n] &= \sum_{m=1-N}^0 \delta[m+n_i] \delta[(m-n)+n_j] \\ &= \delta[(-n_i)-n+n_j] \\ &= \delta[n-(n_j-n_i)] \\ &= \delta_{d_{ij}}. \end{aligned} \quad (13)$$

This means that each row of  $\mathbf{R}_c$  contains a single 1 and the rest of its elements are 0, so it suffices to remind the differences  $d_{ij}$ . If we keep all these differences in the set

$$D = \{d_{ij} = |n_i - n_j|; , n_i, n_j \in S\},$$

the authors of [2] claim that we should obtain  $D \supset [0, N-1]$ , or at least, it suffices to achieve  $D \supset [0, \lfloor N/2 \rfloor]$ . The reason is the following: if  $D$  generates all the numbers  $d \in [0, \lfloor N/2 \rfloor]$ ,

<sup>1</sup>The result in (13) is slightly different from [2, eq. (34)], where  $r_{c_i, c_j}[n] = \delta[n - (n_i - n_j)]$ .

then  $N - d$  generates all the numbers in  $[\lfloor N/2 \rfloor, N]$ . But we notice that the conditions of [2] can be weakened: we assure that it suffices to obtain a set of differences  $D$  such that every number in  $[0, N - 1]$  can be written either as  $d \in D$  or as  $N - d$  (where  $d \in D$ ). In other words, we obtain this weaker condition:

$$D \cup \{N - d, d \in D\} \supset [0, N - 1].$$

This is equivalent to the definition of a circular sparse ruler given in [9].

With this in mind, in this paper we design a general sampling pattern which yields such a condition for  $N = 4^n - 1$ .

### 3. A CLASS OF CIRCULAR SPARSE RULERS

Let us define  $N_1 = 2^n + 1$ ,  $N_2 = 2^n - 1$  which are odd numbers, such that

$$N_1 N_2 = (2^n + 1)(2^n - 1) = 4^n - 1 = N.$$

Moreover,  $N_1$  and  $N_2$  are coprime:  $N_1 - N_2 = 2$  which means that their greatest common divisor is either 1 or 2, but the latter is impossible because they are odd numbers.

We will also define  $p = 2^n$ . As  $N_1 - p = 1 = p - N_2$ , for the same reason,  $p$  turns out to be coprime with  $N_1$  and with  $N_2$ .

The sampling pattern is defined as follows:

- On one hand, we consider the multiples of  $N_1$  which are smaller than  $\lfloor \frac{N}{2} \rfloor = 2^{2n-1} - 1$ , say,

$$0, N_1, 2N_1, \dots, \left\lfloor \frac{N_2}{2} \right\rfloor N_1$$

because

$$\left\lfloor \frac{N_2}{2} \right\rfloor N_1 = (2^{n-1} - 1)(2^n + 1) = 2^{2n-1} - 2^{n-1} - 1.$$

- On the other hand, we also keep the corresponding multiples of  $N_2$ ,

$$N_2, 2N_2, \dots, \left\lfloor \frac{N_1}{2} \right\rfloor N_2$$

which are also smaller than  $\lfloor \frac{N}{2} \rfloor = 2^{2n-1} - 1$  because

$$\left\lfloor \frac{N_1}{2} \right\rfloor N_2 = 2^{n-1}(2^n - 1) = 2^{2n-1} - 2^{n-1}.$$

- Finally, we will also consider some multiples of  $p = 2^n$

$$p, 2p, 3p, \dots, (2^{n-1} - 1)p.$$

But in this case we will keep the samples of the kind

$$N - p, N - 2p, 3p, \dots, N - (2^{n-1} - 1)p$$

which have values greater than  $\lfloor N/2 \rfloor$ .

In summary, we propose the sampling pattern

$$S = S_1 \cup S_2 \cup S_3$$

where

$$S_1 = \{mN_1, m = 0, \dots, 2^{n-1} - 1\} \quad (14)$$

$$S_2 = \{sN_2, s = 1, \dots, 2^{n-1}\}$$

$$S_3 = \{N - lp, l = 1, \dots, 2^{n-1} - 1\}.$$

Notice that those numbers are different; in fact,  $mN_1 \neq sN_2$  because  $N_1$  and  $N_2$  are coprime and  $m < N_2$ ,  $0 < s < N_1$ . Besides,  $mN_1 \neq N - lp \neq sN_2$  because

$$\max(mN_1, sN_2) < N/2 < N - lp.$$

Hence, the cardinality of the set  $S$  of samples is exactly equal to the sum of the cardinalities of the sample sets  $S_1, S_2, S_3$  :

$$M = 2^{n-1} + 2^{n-1} + 2^{n-1} - 1 = 3 \cdot 2^{n-1} - 1$$

Our aim is to prove that the differences  $D$  of the set  $S$ , and its reciprocal set of differences  $D' = \{N - d, d \in D\}$  verify

$$D \cup D' \supset [0, N - 1]. \quad (15)$$

**Theorem 1** *Let  $N = 4^n - 1$ , and let us consider the numbers*

$$N_1 = 2^n + 1, \quad N_2 = 2^n - 1, \quad p = 2^n.$$

*Then the sampling pattern*

$$\begin{aligned} S &= S_1 \cup S_2 \cup S_3 = \\ &= \{mN_1, m = 0, \dots, 2^{n-1} - 1\} \cup \\ &\quad \{sN_2, s = 1, \dots, 2^{n-1}\} \cup \\ &\quad \{N - lp, l = 1, \dots, 2^{n-1} - 1\}, \end{aligned}$$

*is a universal power spectrum estimation pattern (of length  $3 \cdot 2^{n-1} - 1$ ) for signals of length  $N$ .*

**Proof.** It suffices to prove that any  $d \in [0, N - 1]$  can be written either as a difference  $d \in D$ , or as  $N - d \in D$ .

With our proposed design, the set of differences  $D$  is

$$D = S_1 \cup S_2 \cup S_3 \cup D_{1,2} \cup D_{1,3} \cup D_{2,3}$$

where  $D_{i,j}$  contains all differences between nonzero samples of the set  $S_i$  and samples of the set  $S_j$  :

$$\begin{aligned} D_{1,2} &= \{mN_1 - sN_2, m = 1, \dots, 2^{n-1} - 1, s = 1, \dots, 2^{n-1}\} \\ D_{1,3} &= \{N - lp - mN_1, l, m = 1, \dots, 2^{n-1} - 1\} \\ D_{2,3} &= \{N - lp - sN_2, l = 1, \dots, 2^{n-1} - 1, s = 1, \dots, 2^{n-1}\}. \end{aligned}$$

On the other hand, we will also consider the reciprocal set of differences  $D' = \{N - d, d \in D\}$  :

$$D' = S'_1 \cup S'_2 \cup S'_3 \cup D'_{1,2} \cup D'_{1,3} \cup D'_{2,3}.$$

Note that some of these reciprocal sets of differences are:

$$\begin{aligned} S'_3 &= \{lp, l = 1, \dots, 2^{n-1} - 1\} \\ D'_{1,3} &= \{lp + mN_1, l, m = 1, \dots, 2^{n-1} - 1\} \\ D'_{2,3} &= \{lp + sN_2, l = 1, \dots, 2^{n-1} - 1, s = 1, \dots, 2^{n-1}\}. \end{aligned}$$

Let us show how every  $d \in [0, N - 1]$  is generated by at least one of those sets of differences  $D \cup D'$ . To this aim, we will use some consequences of Bezout's identity: as  $N_1$  and  $N_2$  are coprime, any integer  $d$  can be written as

$$d = mN_1 - sN_2 \quad (m, s \in \mathbb{Z}).$$

But we only have  $m = 0, \dots, 2^{n-1} - 1, s = 1, \dots, 2^{n-1}$ . So the differences  $D_{1,2}$  can generate any  $d \in [0, 2^{2n-1} - 2^{n-1}]$  except for 4 types of numbers:

1. the numbers  $d$  which are multiples of  $p = 2^n$
2. the numbers  $l2^n + mN_1$  (but they belong to  $D'_{1,3}$ )
3. the numbers  $l2^n + sN_2$  (but they belong to  $D'_{2,3}$ )
4. the numbers  $l2^n + mN_1 + sN_2$ .

Let us explain why these 4 kinds of numbers do not belong to  $D_{1,2}$ . For the multiples of  $p = 2^n$ , the diofantic equation

$$l2^n = N_1m - N_2s$$

has a solution  $m = s = l2^{n-1}$  since

$$N_1(l2^{n-1}) - N_2(l2^{n-1}) = (N_1 - N_2)l2^{n-1} = l2^n,$$

but those values of  $m, s$  are out of range for any  $p$ . Nevertheless, the general solution of the diofantic equation is

$$l2^n = N_1(l2^{n-1} - N_2r) - N_2(l2^{n-1} - N_1r).$$

Let us prove that there does not exist any  $r, l$  such that those  $m, s$  belong to the corresponding intervals:

$$\begin{aligned} 0 &\leq m = l2^{n-1} - N_2r \leq 2^{n-1} - 1 \\ 1 &\leq s = l2^{n-1} - N_1r \leq 2^{n-1} \end{aligned}$$

In effect,

$$\begin{aligned} m &= l2^{n-1} - (2^{n-1} - 1)r = (l - 1)2^{n-1} + r \\ s &= l2^{n-1} - (2^{n-1} + 1)r = (l - 1)2^{n-1} - r \end{aligned}$$

are out of range; even in the case  $l = 1, m = r = -s$  is impossible since  $m \geq 0, s > 0$ .

As a conclusion, the multiples of  $p = 2^n$  cannot be written as  $N_1m - N_2s$  (but they belong to  $S'_3$ ). The same happens for  $lp + N_1m$ , for  $lp + N_2s$  and  $l2^n + mN_1 + sN_2$ . In other words, these numbers do not belong to  $D_{1,2}$ . This is the reason why we had to include the samples  $S_3$  in the definition of our set  $S$ .

In order to generate the numbers  $l2^n + mN_1 + sN_2$ , let us write them as  $(l + m + s)2^n + (m - s)$ :

- if  $m \neq s$ , it is generated by  $D_{1,2}$  because it is not a multiple of  $2^n$ .
- If  $m = s$ , this number is  $(l + 2)2^n$  which is a multiple of  $p$ , so it belongs to  $S_3$  if  $l \leq 2^{n-1} - 3$ . Otherwise,
  - If  $l = 2^{n-1} - 2$  the number is  $2^{2n-1} = \lfloor N/2 \rfloor + 1$  which will be generated below by  $D_{2,3}$ .
  - if  $l = 2^{n-1} - 1$  the number is  $d = 2^{2n-1} + 2^n$  but  $N - d = 2^{2n-1} - 2^n - 1$  which has already been generated by  $D_{1,2}$ .

So far, we have seen how to generate any difference  $d \in [0, 2^{2n-1} - 2^{n-1}]$ . It remains to prove that we can also generate any  $d \in [2^{2n-1} - 2^{n-1}, \lfloor N/2 \rfloor]$ . Notice that

$$(2^{n-1} - s)p + sN_2 = (2^{n-1})p - s = 2^{2n-1} - s.$$

This means that these numbers belong to  $D'_{2,3}$ . Hence, for  $s = 1, \dots, 2^{n-1} - 1$  we obtain all values of

$$2^{2n-1} - 2^{n-1} + 1 \leq d \leq 2^{2n-1} - 1 = \lfloor N/2 \rfloor$$

that we had not been able to generate so far; now we have guaranteed that they belong to  $D'_{2,3}$ .

This way, we have generated all numbers  $d \in [0, \lfloor N/2 \rfloor]$ . The rest of the differences  $(N - d)$  will generate the set  $[\lfloor N/2 \rfloor + 1, N]$ . In this way we have finally derived the condition that  $D \cup D' \supset [0, N - 1]$  of Equation (15).

All these facts conclude the desired proof. ■

#### 4. NUMERICAL EXAMPLES

*Example 1:* For  $N = 15 = 4^2 - 1$ , we get the sampling pattern made by multiples of  $N_2 = 2^2 - 1 = 3$ ,  $N_1 = 2^2 + 1 = 5$ , and 15 – a multiple of 4 :

$$S = \{0, 3, 6, 5, 11\}.$$

There are only  $M = 5$  samples, but their differences  $d$  are

$$D = \{0, 3, 6, 5, 11, 2, 8, 1\}$$

and  $N - d = 15 - d$  provide

$$D' = \{15, 12, 9, 10, 4, 13, 7, 14\}$$

so we finally have generated all numbers in  $[0, 15]$  since  $D \cup D' = [0, 15]$ .

Recall that the block matrix

$$\begin{bmatrix} \mathbf{R}_c[0] \\ \mathbf{R}_c[1] \end{bmatrix},$$

made of  $N = 15$  columns, will have full column rank. For the sake of clarity, we can build these matrices: most of their elements are 0, but we know that there is at least a 1 in the

$(d + 1)$ -th column of  $\mathbf{R}_c[0]$  (indexed by the differences  $d \in D$ ):

$$\mathbf{R}_c[0] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and, for each  $d \in D$ , there is (by symmetry) at least a 1 in the  $(N - d + 1)$ -th column of  $\mathbf{R}_c[1]$ , indexed by  $d' + 1$ , being  $d' \in D' \setminus \{N\}$ . In other words, in this case we obtain 1's in the columns indexed as  $(16 - d)$  where  $d \in D \setminus \{0\}$ .

$$\mathbf{R}_c[1] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So it is straightforward to observe that the  $16 \times 15$  matrix

$$\begin{bmatrix} \mathbf{R}_c[0] \\ \mathbf{R}_c[1] \end{bmatrix}$$

has at least a 1 in each column, so it has full column rank, as desired.

*Example 2:* For  $N = 1023 = 4^5 - 1$ , the sampling pattern is made by 16 multiples of  $N_1 = 2^5 + 1 = 33$ , by 16 multiples of  $N_2 = 2^5 - 1 = 31$ , and by 15 multiples of 32 (that will be subtracted from  $N = 1023$ ). We have a total amount of  $M = 47$  samples whose differences actually generate all 1023 numbers in  $[0, 1022]$ .

For the designed patterns the compression rate is

$$\frac{M}{N} = \frac{3(2^{n-1}) - 1}{4^n - 1} \approx \frac{3 \cdot 2^{n-1}}{4^n} = \frac{3}{2^{n+1}}.$$

Notice that this approximated compression rate is halved whenever we increase  $n$ . For instance, for  $n = 3$  ( $N = 63$ ) we keep only  $M = 11$  samples, so the real compression rate is  $11/63 = 17,46\%$ , which is smaller than the estimated compression rate  $3/16 = 18,75\%$ . For  $n = 4$  we have an approximated compression rate of  $9,375\%$  for  $N = 255$  (but we only need  $M = 23$  samples, so the compression rate is actually  $23/255 = 9,01\%$ ).

In any case, we have a good approximation of the compression rate, and for  $n = 5, 6, 7$  we get  $4,68\%$ ,  $2,34\%$  and  $1,17\%$  for, respectively,  $N = 1023$ ,  $N = 4095$  and  $N = 16383$ .

## 5. CONCLUSION

We have designed a family of circular sparse rulers suitable for reconstruction of the power spectrum at sampling rates significantly below the Nyquist rate. These rulers can be seen as multicoset patterns implemented by a simple coprime sampling device. The required pattern, even if it is long, is easily designed once its length is specified, which constitutes an interesting advantage over minimal sparse rulers.

## 6. REFERENCES

- [1] G. Leus and D. D. Ariananda, "Power spectrum blind sampling," *IEEE Signal Process. Lett.*, vol. 18, no. 8, pp. 443–446, Aug. 2011.
- [2] D. D. Ariananda and G. Leus, "Compressive wideband power spectrum estimation," *IEEE Trans. Signal Process.*, vol. 60, no. 9, pp. 4775–4789, Sept. 2012.
- [3] E. Axell, G. Leus, E. G. Larsson, and H. V. Poor, "Spectrum sensing for cognitive radio : State-of-the-art and recent advances," *IEEE Signal Process. Mag.*, vol. 29, no. 3, pp. 101–116, May 2012.
- [4] D. L. Donoho, "Compressed sensing," *IEEE Trans. Inf. Theory*, vol. 52, no. 2, pp. 1289–1306, Apr. 2006.
- [5] Emmanuel J. Cands and Michael B. Wakin, "An introduction to compressive sampling," *Signal Processing Magazine, IEEE*, vol. 25, pp. 21–30, March 2008.
- [6] M. Mishali and Y. C. Eldar, "Blind multiband signal reconstruction: Compressed sensing for analog signals," *IEEE Trans. Signal Process.*, vol. 57, pp. 993–1009, Mar. 2009.
- [7] M.E. Domínguez-Jiménez and N. González-Prelcic, "Design of universal multicoset sampling patterns for sensing of compressed sensing of multiband sparse signals," in *IEEE Int. Conf. Acoust., Speech, Signal Process. (ICASSP)*, 2012.
- [8] J. Leech, "On the representation of  $1, 2, \dots, n$  by differences," *Journal of the London Mathematical Society*, vol. 1, pp. 160–169, 1956.
- [9] D. Romero and G. Leus, "Compressive Covariance Sampling," in *Inform. Theory Appl. Workshop*, Feb. 2013.
- [10] P.P. Vaidyanathan and P. Pal, "Theory of sparse coprime sensing in multiple dimensions," *IEEE Trans. Signal Process.*, vol. 59, pp. 3592–3608, Aug. 2011.
- [11] P.P. Vaidyanathan and P. Pal, "Sparse sensing with coprime samplers and arrays," *IEEE Trans. Signal Process.*, vol. 59, pp. 573–586, Feb. 2011.