DIAMOND CONTOUR-BASED PHASE RECOVERY FOR (CROSS)-QAM CONSTELLATIONS

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ABSTRACT

Two new iterative methods for blind phase estimation of QAM signals are presented. The algorithms seek the maxima of certain cost functions derived from the dispersion of the de-rotated data with respect to a diamond-shaped contour, and their distinctive feature is the inclusion of a sign nonlinearity at every iteration. A connection between one of these methods and the standard fourth-power method is presented. Simulations with cross-QAM constellations show that, when properly initialized, the new schemes present lower variance than the fourth-power method, and may even outperform Cartwright's eighth-order method.

Keywords: Carrier phase estimation, QAM constellations, synchronization.

1. INTRODUCTION

We consider the problem of blind carrier phase acquisition in QAM digital communication systems. The detected data are assumed to be of the form

$$r_k = a_k e^{j\theta} + n_k, \qquad k = 0, 1, \ldots, L-1, \tag{1}$$

where a_k is the kth transmitted symbol, drawn equiprobably from a QAM constellation, and n_k is the complex-valued kth noise sample. The noise is assumed white Gaussian and circular with variance σ^2, and independent of the symbols. The goal is to identify θ without knowledge of the symbols a_k.

A classical approach to this problem results in the fourth-power method,

$$\hat{\theta}_{4P} = \frac{1}{4} \arg\left\{ -\sum_{k=0}^{L-1} r_k^4 \right\}. \tag{2}$$

The performance of the fourth-power method applied to square QAM constellations is considered acceptable (it is known to be the Maximum-Likelihood estimator as the SNR goes to zero [1]), while it is poor for cross QAM constellations [2]. Thus, alternatives have been devised to improve phase detection for such constellations. Among them, one approach is to use higher order statistics of the observed signal. In [3], Cartwright presents a method using eighth-order statistics with improved performance with respect to the fourth-power method. The price paid in Cartwright's method (further referred to as C-VIII) is the higher computational cost.

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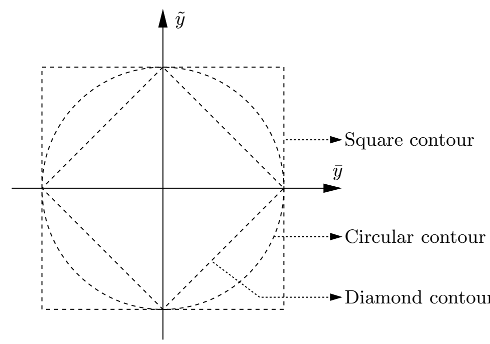


Fig. 1. Possible contours.

Nevertheless, other approaches are possible. For example, a joint blind adaptive equalization and phase recovery was presented in [4] based on the minimization of the dispersion of the equalizer output with respect to a square contour. This idea can be used to devise blind phase estimators. In principle, different contours could be considered; Figure 1 illustrates three possibilities for which the following conditions and related cost functions can be written:

- Circular contour (CM - *Constant Modulus*):

$$|y_k|^2 = \bar{y}_k^2 + \tilde{y}_k^2 = \text{constant} \tag{3}$$

$$J_{\text{CM}} \doteq \mathrm{E}\left[\left(|y_k|^2 - \gamma_{\text{CM}} \right)^2 \right]; \tag{4}$$

- Square contour (SC):

$$\max(|\bar{y}_k|, |\tilde{y}_k|) = \frac{|\bar{y}_k - \tilde{y}_k| + |\bar{y}_k + \tilde{y}_k|}{2} = \text{constant} \tag{5}$$

$$J_{\text{SC}} \doteq \mathrm{E}\left[\left(|\bar{y}_k - \tilde{y}_k| + |\bar{y}_k + \tilde{y}_k| - \gamma_{\text{SC}} \right)^2 \right]; \tag{6}$$

- Diamond contour (DC):

$$|\bar{y}_k| + |\tilde{y}_k| = \text{constant} \tag{7}$$

$$J_{\text{DC}} \doteq \mathrm{E}\left[\left(|\bar{y}_k| + |\tilde{y}_k| - \gamma_{\text{DC}} \right)^2 \right], \tag{8}$$

where $y_k = \bar{y}_k + j\tilde{y}_k$ are the equalizer outputs, and γ_{CM}, γ_{SC} and γ_{DC} are appropriate constants.

Note that the CM cost is not useful for phase estimation purposes since it is insensitive to phase rotations. The square contour, on the other hand, seems appropriate for QAM constellations. Geometrically, we see that such minimization should be equivalent (for the phase estimation problem) to maximizing the dispersion with respect to the diamond contour, i.e., maximizing $J_{\rm DC}$. In this paper, we focus on the cost $J_{\rm DC}$ so as to derive practical phase recovery algorithms, which we will call *Diamond Contour Algorithms* (DCAs).

The paper is structured as follows. In section 2, starting from $J_{\rm DC}$ we derivate two new iterative algorithms for blind phase recovery of QAM signals. Section 3 contains a theoretical analysis of one of the algorithms, which relates it to the fourth-power estimator. In section 4 we evaluate the computational costs of the newly developed algorithms, and compare them to Cartwright's eighth-order method. Section 5 presents simulation results showing the performance of the considered methods. Finally, in section 6 we draw the main conclusions of our research.

2. ALGORITHM DERIVATION

We assume the model (1) and define the de-rotated observations

$$y_k = \mathrm{e}^{-j\hat{\theta}} r_k, \tag{9}$$

where $\hat{\theta}$ is the phase estimate. Given the blind nature of the estimate and the quadrant symmetry of the QAM constellation, we can only expect to identify angles within a 90° range (In fact, all the above cost functions exhibit 90° periodicity). Hence we assume $-\frac{\pi}{4} \leq \hat{\theta} < \frac{\pi}{4}$. Considering (8), observe that in

$$(|\bar{y}_k| + |\tilde{y}_k| - \gamma_{\rm DC})^2 = \left[|\bar{y}_k|^2 + |\tilde{y}_k|^2 - \gamma_{\rm DC}^2\right] + 2|\bar{y}_k||\tilde{y}_k| - 2\gamma_{\rm DC}(|\bar{y}_k| + |\tilde{y}_k|)$$

the bracketed term does not depend on $\hat{\theta}$. Hence, aside from a constant term, the cost $J_{\rm DC}$ is the (weighted) difference of two other costs, J_a and J_b, defined as:

$$\begin{aligned} J_{\rm DC}(\hat{\theta}) &= 2J_a(\hat{\theta}) - 2\gamma_{\rm DC} J_b(\hat{\theta}) + \text{constant} && (10)\\ J_a(\hat{\theta}) &\doteq \mathrm{E}\left[|\bar{y}_k| \cdot |\tilde{y}_k|\right] && (11)\\ J_b(\hat{\theta}) &\doteq \mathrm{E}\left[|\bar{y}_k| + |\tilde{y}_k|\right]. && (12) \end{aligned}$$

It has been checked numerically that, for QAM constellations, both J_a and J_b have their global maxima at $\hat{\theta} = \theta$. Hence, it makes sense to consider phase estimators departing from the maximization of either J_a or J_b. This we do next.

2.1. Derivation of the algorithm from the cost $J_a(\hat{\theta})$

If we write $r_k = \bar{r}_k + j\tilde{r}_k$, then from (9)

$$\bar{y}_k = \bar{r}_k \cos\hat{\theta} + \tilde{r}_k \sin\hat{\theta}, \quad \tilde{y}_k = \tilde{r}_k \cos\hat{\theta} - \bar{r}_k \sin\hat{\theta}. \tag{13}$$

Using basic trigonometric relationships, it is found that

$$\bar{y}_k \cdot \tilde{y}_k = \bar{r}_k\tilde{r}_k \cos(2\hat{\theta}) - \frac{1}{2}(\bar{r}_k^2 - \tilde{r}_k^2)\sin(2\hat{\theta}). \tag{14}$$

Let us look at the stationary points of $J_a(\hat{\theta})$. Differentiating the absolute value of (14) and taking expectations,

$$\begin{aligned} \frac{\partial J_a(\hat{\theta})}{\partial\hat{\theta}} &= -\mathrm{E}\left[\text{sign}\left(\bar{y}_k\tilde{y}_k\right)\ (\bar{r}_k^2 - \tilde{r}_k^2)\right]\cos(2\hat{\theta}) \\ &\quad -\mathrm{E}\left[\text{sign}\left(\bar{y}_k\tilde{y}_k\right) 2\,\bar{r}_k\tilde{r}_k\right]\sin(2\hat{\theta}). && (15) \end{aligned}$$

Observe that

$$\begin{aligned} \bar{r}_k^2 - \tilde{r}_k^2 &= \mathfrak{Re}\left\{r_k^2\right\}, \quad 2\,\bar{r}_k\tilde{r}_k = \mathfrak{Im}\left\{r_k^2\right\}, && (16)\\ \text{sign}\left(\bar{y}_k\tilde{y}_k\right) &= \text{sign}\left(2\,\bar{y}_k\tilde{y}_k\right) \\ &= \text{sign}\left(\mathfrak{Im}\left\{r_k^2 \mathrm{e}^{-j2\hat{\theta}}\right\}\right) \doteq s_k(\hat{\theta}). && (17) \end{aligned}$$

At any stationary point $\hat{\theta}_*$ it must hold $\partial J_a(\hat{\theta}_*)/\partial\hat{\theta} = 0$, i.e.,

$$\begin{aligned} \tan(2\hat{\theta}_*) &= -\frac{\mathrm{E}\left[\ \mathfrak{Re}\left\{r_k^2\right\} s_k(\hat{\theta}_*)\right]}{\mathrm{E}\left[\ \mathfrak{Im}\left\{r_k^2\right\} s_k(\hat{\theta}_*)\right]} && (18)\\ &= \tan\left(\arg\left\{-j\mathrm{E}\left[r_k^2\ s_k(\hat{\theta}_*)\right]\right\}\right). && (19) \end{aligned}$$

Expression (19) suggests an iterative method to solve for $\hat{\theta}_*$ off-line. Given the observations r_k, $k = 0, \ldots, L-1$, and given a suitable initialization $\hat{\theta}_0$, $\hat{\theta}_*$ can be approximated by

$$\begin{aligned} s_k(\hat{\theta}_n) &= \text{sign}\left(\mathfrak{Im}\left\{r_k^2 \mathrm{e}^{-j2\hat{\theta}_n}\right\}\right),\ k = 0, \ldots, L-1, && (20)\\ \hat{\theta}_{n+1} &= \frac{1}{2}\arg\left\{\sum_{k=0}^{L-1}\left[r_k^2 s_k(\hat{\theta}_n)\right]\right\} - \frac{\pi}{4}, && (21) \end{aligned}$$

for $n = 1, 2, \ldots$ We will refer to algorithm (20)-(21) as DCA-a.

2.2. Derivation of the algorithm from the cost $J_b(\hat{\theta})$

We can follow the same approach of the previous subsection for $J_b(\hat{\theta})$. Using (13) we can differentiate (12):

$$\begin{aligned} \frac{\partial J_b(\hat{\theta})}{\partial\hat{\theta}} &= \mathrm{E}\left[\frac{\partial|\bar{y}_k|}{\partial\hat{\theta}} + \frac{\partial|\tilde{y}_k|}{\partial\hat{\theta}}\right] && (22)\\ &= \mathrm{E}\left[\text{sign}\left(\bar{y}_k\right)\tilde{y}_k - \text{sign}\left(\tilde{y}_k\right)\bar{y}_k\right]. && (23) \end{aligned}$$

At a stationary point $\hat{\theta}_*$, (23) equals zero, so that

$$\mathrm{E}\left[\text{sign}\left(\bar{y}_k\right)\tilde{y}_k\right] = \mathrm{E}\left[\text{sign}\left(\tilde{y}_k\right)\bar{y}_k\right] \tag{24}$$

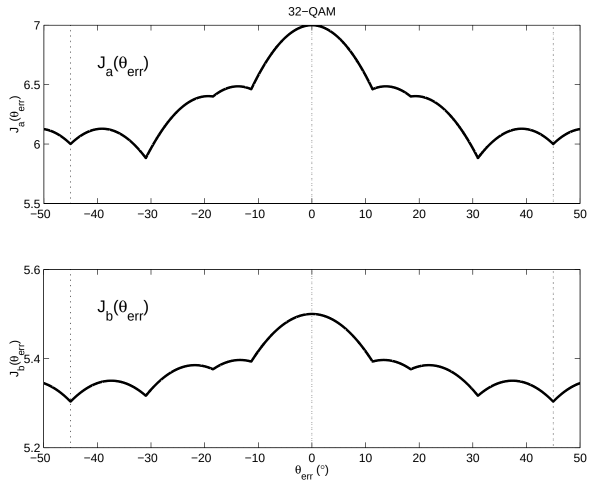
holds. Now, using again (13) we arrive at:

$$\begin{aligned} \tan\hat{\theta}_* &= \frac{\mathrm{E}\left[\text{sign}\left(\bar{y}_k\right)\tilde{r}_k - \text{sign}\left(\tilde{y}_k\right)\bar{r}_k\right]}{\mathrm{E}\left[\text{sign}\left(\bar{y}_k\right)\bar{r}_k + \text{sign}\left(\tilde{y}_k\right)\tilde{r}_k\right]} && (25)\\ &= \frac{-\mathrm{E}\left[\ \mathfrak{Im}\left\{\text{csign}\left(y_k\right)\cdot r_k^*\right\}\right]}{\mathrm{E}\left[\ \mathfrak{Re}\left\{\text{csign}\left(y_k\right)\cdot r_k^*\right\}\right]} && (26)\\ &= \tan\left(-\arg\left\{\mathrm{E}\left[\text{csign}\left(y_k\right)\cdot r_k^*\right]\right\}\right). && (27) \end{aligned}$$

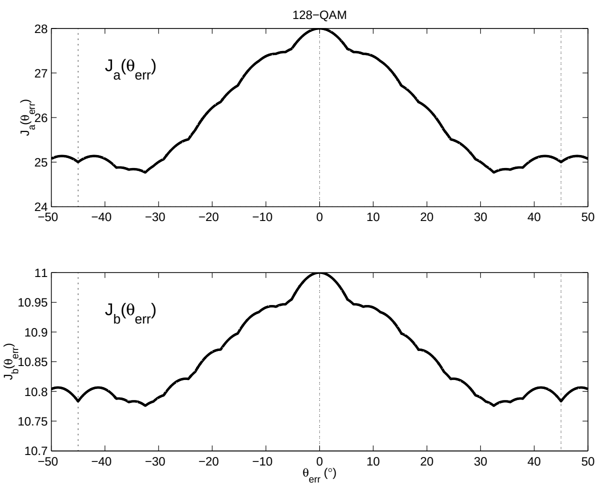
where $\text{csign}\,(z) \doteq \text{sign}\,(\bar{z}) + j\text{sign}\,(\tilde{z})$, being $z = \bar{z} + j\tilde{z}$. Again, the condition (27) suggests an iterative estimation procedure. Given the observations r_k, $k = 0, \ldots, L-1$, and a suitable initialization $\hat{\theta}_0$, $\hat{\theta}_*$ can be approximated by

$$\hat{\theta}_{n+1} = -\arg\left\{\sum_{k=0}^{L-1}\text{csign}\left(r_k \cdot \mathrm{e}^{-j\hat{\theta}_n}\right)\cdot r_k^*\right\}, \tag{28}$$

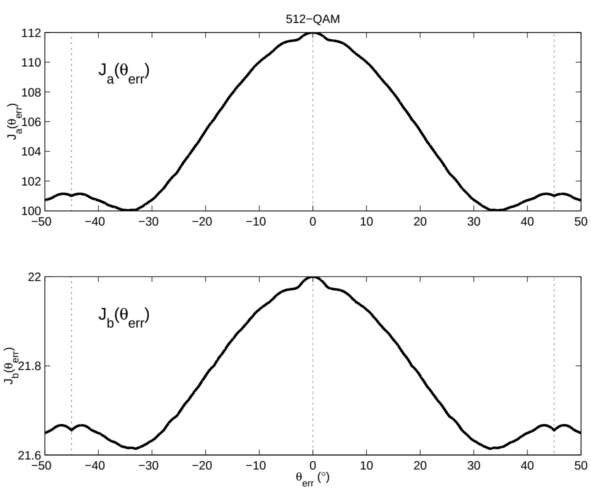
for $n = 1, 2, \ldots$ We will refer to algorithm (28) as DCA-b.



(a) 32-QAM



(b) 128-QAM



(c) 512-QAM

Fig. 2. The costs J_a (upper plots) and J_b (lower plots) in terms of the phase error $\tilde{\theta}$ for the cross-constellations 32-QAM, 128-QAM and 512-QAM.

3. DISCUSSION

3.1. Algorithm initialization

In general, the costs J_a, J_b are multimodal functions of $\hat{\theta}$ due to the hard sign nonlinearities involved. For example, Figures 2(a), 2(b) and 2(c) show plots of these costs in terms of the phase error $\tilde{\theta} = \theta - \hat{\theta}$ for 32-, 128- and 512-QAM cross constellations. Hence, the initialization of the iterative estimators must be good enough so that $\hat{\theta}_0$ lies within the domain of attraction of the global maximum, which seems to be located at the desired point $\tilde{\theta} = 0$. Possible initialization options include the monomial-based estimators of [5], which for $m = 0, \ldots, 4$ are given by

$$\hat{\theta}^{[m]} \doteq \frac{1}{4} \arg \left\{ \mathrm{E} \left[- \sum_{k=0}^{L-1} |r_k|^m \mathrm{e}^{j4 \arg\{r_k\}} \right] \right\}. \qquad (29)$$

(Note that $\hat{\theta}^{[4]}$ reduces to $\hat{\theta}_{4\mathrm{P}}$.) Though the quality of these estimators is poor for cross-QAM constellations, it is usually good enough for locating the initial phase in the right region. From an implementation point of view, the estimators $\hat{\theta}^{[2]}$ or $\hat{\theta}^{[4]}$ seem suitable for initialization of DCA-a, while $\hat{\theta}^{[0]}$ seems appropriate for DCA-b.

It is also worth mentioning that for dense square QAM constellations (64-QAM and higher), the costs J_a and J_b appear to be unimodal, as can be seen in figures 3(a),3(b) and 3(c). This fact would relax the initialization requirements for the DCA estimators, when applied to such square constellations.

3.2. Convergence of "soft" DCA-a

As we can see, both algorithms (20)-(21) and (28) present hard nonlinearities in form of $\mathrm{sign}\,(\cdot)$ functions, which make convergence analysis a difficult task. It is nevertheless instructive to investigate the behavior of the "soft" iterations that result if the $\mathrm{sign}\,(\cdot)$ functions were eliminated. With this in mind, and without loss of generality, we will substitute the summations involved by expectations. Doing so for (28), we obtain the soft iteration

$$\hat{\theta}_{n+1} = -\arg \left\{ \mathrm{e}^{-j\hat{\theta}_n} \mathrm{E} \left[|r_k|^2 \right] \right\} = \hat{\theta}_n, \qquad (30)$$

which is clearly not useful.

On the other hand, the soft iteration derived from (21) is

$$\hat{\theta}_{n+1} = \frac{1}{2} \arg \left\{ \mathrm{E} \left[r_k^2 \cdot \mathfrak{Im} \left\{ r_k^2 \mathrm{e}^{-j2\hat{\theta}_n} \right\} \right] \right\} - \frac{\pi}{4}. \qquad (31)$$

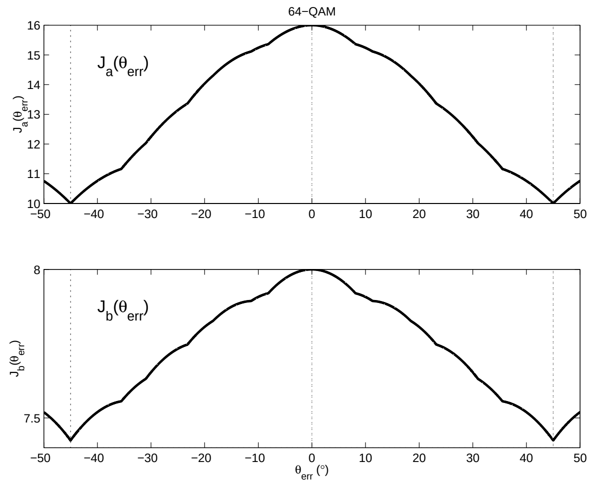
The analysis of (31) offers interesting results, which we condensate in the following theorem.

Theorem 3.1 *The fixed points of the soft iteration* (31) *are given by*

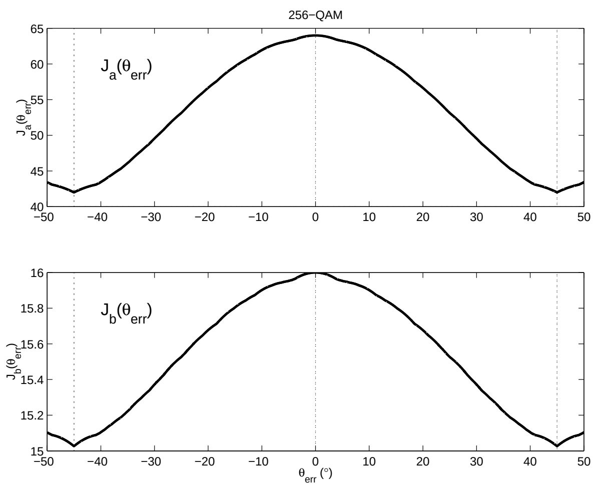
$$\hat{\theta}_*^{(m)} = \frac{1}{4} \Big(\mathit{arg} \left\{ \mathrm{E} \left[r_k^4 \right] \right\} + m\pi \Big), \quad m \in \mathbb{Z}, \qquad (32)$$

and the only stable fixed points are those given by odd values of m *in* (32).

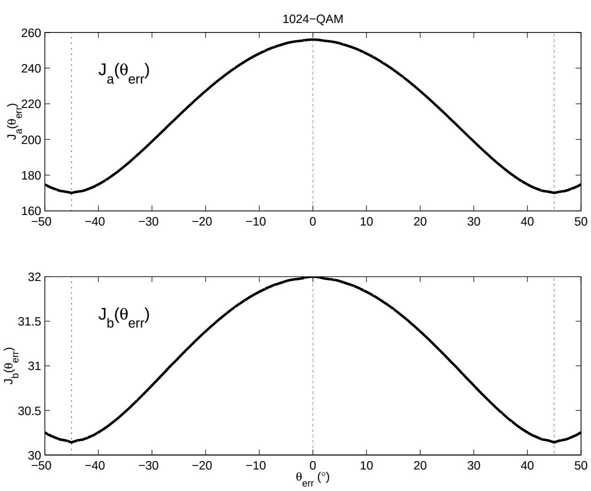
The proof is given in Appendix A. Observe that the fourth-power estimate corresponds with the odd values of m, that is, only the fourth-power estimate is the only stable point of the soft iteration.



(a) 64-QAM



(b) 256-QAM



(c) 1024-QAM

Fig. 3. The costs J_a (upper plots) and J_b (lower plots) in terms of the phase error $\tilde{\theta}$ for the square constellations 64-QAM, 256-QAM and 1024-QAM.

	C-VIII	**4th power**	**DCA-a**	**DCA-b**
$+$	$8L$	$4L$	$3L$	$4L$
$\times$	$11L$	$6L$	$2L$	$4L$

Table 1. Computational burden of the compared methods. For the iterative methods, the cost is given in operations per iteration and does not include initialization costs. ($+$): real additions; ($\times$): real products.

3.3. Effect of the noise

In the derivation of the iterative phase estimators the effects of noise have been ignored. Extensive simulations have shown that, when suitably initialized, DCA-a and DCA-b do not present any noticeable bias even when noise is present. A formal proof is not available at this time though.

4. IMPLEMENTATION ISSUES

In order to make a fair comparison between different estimators, their corresponding computational costs must be determined. Table 1 lists the approximate complexities of the compared algorithms in terms of real additions and products. L is the number of observations. Note that C-VIII is about twice as expensive as the fourth-power estimate. For example, if the fourth-power method is used to start up and then two iterations of DCA-a are performed, the complexity of the resulting estimator is below of that of C-VIII. On the other hand, the estimate $\hat{\theta}^{[0]}$ given by (29) requires L phase extractions and $2L$ real additions, and if it is used to initialize DCA-b, then the iterations can be performed without any multiplication. Thus, depending on the cost assigned to the phase extraction operation, this approach may be competitive.

In addition, the DCA methods have other advantages over C-VIII. The latter requires eighth-order products of the received data, which may cause overflow problems in fixed-point arithmetic implementations. On the other hand, DCA-a only uses second-order products and phase rotations, and DCA-b only phase rotations, which are numerically more stable than power operations. Also, phase rotations can be efficiently performed using CORDIC hardware implementations (cf. [6]).

5. SIMULATION RESULTS

Based on the discussion in the previous section, for our simulations we set the maximum number of iterations for DCA-a and DCA-b to three and two respectively. Both methods are initialized using $\hat{\theta}_0 = \hat{\theta}_{4\mathrm{P}}$.

Figures 4 and 5 show the performance of the proposed estimators, as a function of the SNR per bit $\mathrm{SNR_b}$, in comparison with the fourth-power and C-VIII methods, using $L = 1000$ and $L = 2000$ observed symbols for the 32-QAM and 128-QAM cases, respectively. The phase shift θ of the received constellation has been set to 15°, though simulations show that the performance is not affected by the choice of this parameter. The performance is measured in terms of Root-Mean Square Error (RMSE) of the phase estimate, averaged over 10000 Monte Carlo simulations. For the 32-QAM case, we can see that the performance of the DCA estimates is comparable to that of C-VIII for $\mathrm{SNR_b}$ above 10 dB. In the case of 128-QAM, DCA-b achieves the performance of C-VIII for $\mathrm{SNR_b} \geq 20$ dB, while DCA-a is slightly worse.

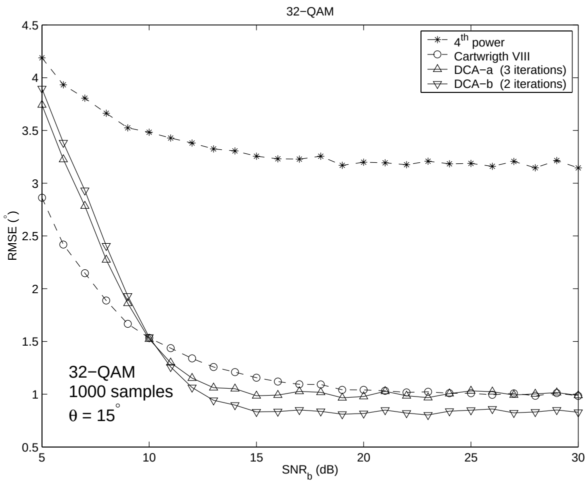


Fig. 4. Results for 32-QAM.

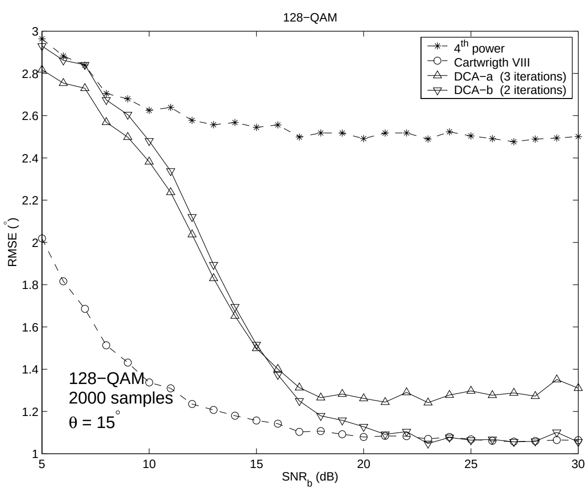


Fig. 5. Results for 128-QAM.

Figures 6 and 7 show the influence of the length of the observation L for a fixed constellation (32-QAM) and fixed SNR_b, which are 10 and 20 dB respectively. It can be observed that for both cases DCA-a and DCA-b reach the performance of C-VIII starting from a given number of symbols L, which is smaller the higher the SNR is. Results for larger cross-constellations are similar, aside from the fact that larger values of L are needed to reach the same performance as in 32-QAM, both for C-VIII and the DCA algorithms.

The RMSE of the DCA estimators with square QAM constellations was also seen to be similar or slightly worse than that of the fourth-power method, so that there seems to be no clear motivation for their use in such scenarios. However, for high order square QAM constellations, the DCA approach may not require accurate initialization, and hence it could be useful as a low-cost, numerically robust alternative to the fourth-power method.

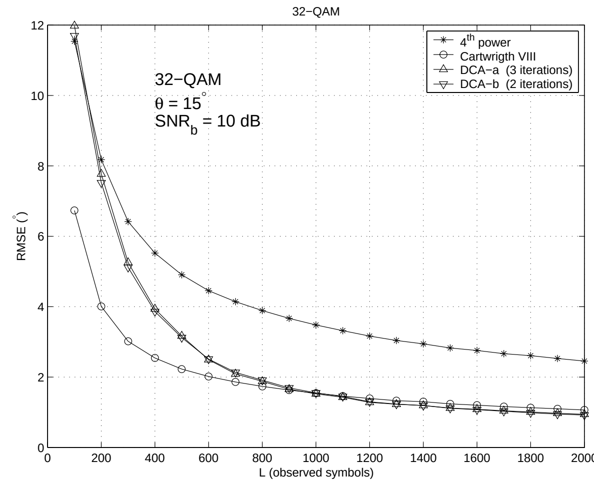


Fig. 6. Influence of L for SNR_b = 10 dB.

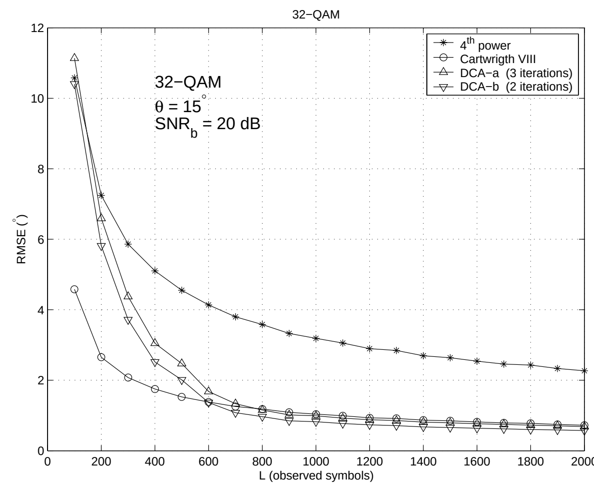


Fig. 7. Influence of L for SNR_b = 20 dB.

6. CONCLUSIONS

Two novel iterative methods for blind phase recovery of QAM signals have been presented. When suitably initialized, their performance with cross-QAM constellations is similar or even better than that of Cartwright's eighth-order method at similar computational cost. Moreover, they avoid potentially troublesome high power operations over the observed data, which helps preventing overflow problems in finite precision implementations. Their behavior with square QAM constellations is close to that of the standard fourth-power estimate. However, for dense square constellations, initialization ceases to be an issue, so that the proposed methods may constitute a low-cost, numerically robust alternative to the fourth-power method.

APPENDIX

A. PROOF OF THEOREM 3.1

Starting from the soft iteration (31), we want to obtain the expression of its fixed points. Let us introduce the following quantities:

$$Z(\hat{\theta}) = \mathrm{E}\left[r_k^2 \cdot \mathfrak{Im}\left\{r_k^2 \mathrm{e}^{-j2\hat{\theta}}\right\}\right] \quad (33)$$

$$A = \mathrm{E}\left[\mathfrak{Re}^2\left\{r_k^2\right\}\right] \quad (34)$$

$$B = \mathrm{E}\left[\mathfrak{Im}^2\left\{r_k^2\right\}\right] \quad (35)$$

$$C = \mathrm{E}\left[\mathfrak{Re}\left\{r_k^2\right\}\ \mathfrak{Im}\left\{r_k^2\right\}\right] \quad (36)$$

$$\rho = \left|\mathrm{E}\left[r_k^4\right]\right|, \quad (37)$$

with $\rho^2 = (A-B)^2 + 4C^2$. Eq. (31) can now be rewritten as

$$0 = \arg\left\{Z(\hat{\theta}_n)\right\} - \left(2\hat{\theta}_{n+1} + \frac{\pi}{2}\right). \quad (38)$$

If the iteration reaches a stationary point, then (38) holds with $\hat{\theta}_{n+1} = \hat{\theta}_n = \hat{\theta}_*$, so that we can write

$$\mathfrak{Re}\left\{Z(\hat{\theta}_*) \cdot \mathrm{e}^{-j(2\hat{\theta}_* + \frac{\pi}{2})}\right\} \geq 0, \quad (39a)$$

$$\mathfrak{Im}\left\{Z(\hat{\theta}_*) \cdot \mathrm{e}^{-j(2\hat{\theta}_* + \frac{\pi}{2})}\right\} = 0. \quad (39b)$$

Now, if we use the following two equalities (where $Z = \bar{Z} + j\tilde{Z}$ is used):

$$\begin{aligned} Z(\hat{\theta}) \cdot \mathrm{e}^{-j(2\hat{\theta}+\frac{\pi}{2})} &= \left[\tilde{Z}(\hat{\theta})\cos(2\hat{\theta}) - \bar{Z}(\hat{\theta})\sin(2\hat{\theta})\right] \\ &\quad -j\left[\bar{Z}(\hat{\theta})\cos(2\hat{\theta}) - \tilde{Z}(\hat{\theta})\sin(2\hat{\theta})\right], \\ Z(\hat{\theta}) &= \left[C\cos(2\hat{\theta}) - A\sin(2\hat{\theta})\right] \\ &\quad +j\left[B\cos(2\hat{\theta}) - C\sin(2\hat{\theta})\right], \end{aligned} \quad (40)$$

after some simple algebraic and trigronometric manipulations, we find that (39b) implies

$$\tan(4\hat{\theta}_*) = \frac{2C}{A-B}. \quad (41)$$

If we notice that $\mathrm{E}\left[r_k^4\right] = (A-B) + j2C$, then solving (41) for $\hat{\theta}_*$ finally yields the desired result shown in (32).

We will now prove that the fourth-power estimate (m odd in (32)) is the only stable fixed point of the soft iteration. In order to check for stability, let us denote the right-hand side of (31) by $F(\hat{\theta}_n)$, so that we can express the iteration as $\hat{\theta}_{n+1} = F(\hat{\theta}_n)$. Then, a fixed point $\hat{\theta}_*^{(m)}$ will be locally convergent if the derivative of F satisfies $\left|F'(\hat{\theta}_*^{(m)})\right| < 1$, and unstable otherwise. We can rewrite (31) as:

$$F(\hat{\theta}) = \frac{1}{2}\arctan\left\{\frac{\tilde{Z}(\hat{\theta})}{\bar{Z}(\hat{\theta})}\right\} - \frac{\pi}{4},$$

so that the derivative of F can be written as

$$F'(\hat{\theta}) = \frac{1}{2|Z(\hat{\theta})|^2}\left[\bar{Z}(\hat{\theta})\tilde{Z}'(\hat{\theta}) - \bar{Z}'(\hat{\theta})\tilde{Z}(\hat{\theta})\right]. \quad (42)$$

Now, using (40), and after some algebraic manipulations in the numerator and denominator of (42), we reach:

$$\begin{aligned} F'(\hat{\theta}) &= \left[AB - C^2\right] / \left[C^2 + B^2\cos^2(2\hat{\theta}) + A^2\sin^2(2\hat{\theta})\right. \\ &\quad \left. -\ C(A+B)\,2\cos(2\hat{\theta})\ \sin(2\hat{\theta})\right]. \end{aligned} \quad (43)$$

In (43), the denominator is positive for being a squared modulus, and the numerator can be proven to be positive since due to the well-known Cauchy-Schwarz's inequality, it holds that $AB \geq C^2$. Thus, $F'(\hat{\theta}) \geq 0$. We want to check whether $F'(\hat{\theta}_*) < 1$ holds for the fixed points of the soft iteration, given by (32). Using (37), it follows that

$$\mathrm{E}\left[r_k^4\right]\mathrm{e}^{-j4\hat{\theta}_*^{(m)}} = (-1)^m \rho.$$

After some straightforward algebraic and trigonometric manipulations, one finds that

$$F'(\hat{\theta}_*^{(m)}) = \frac{AB - C^2}{(AB - C^2) + \frac{1}{2}\rho^2 - \frac{A+B}{2}(-1)^m \rho},$$

where the first term in the denominator is equal to the numerator. This causes the test for $F'(\hat{\theta}_*) < 1$ to be equivalent to

$$\rho^2 > (A+B)\ (-1)^m \rho.$$

Recalling that $\rho^2 = (A-B)^2 + 4C^2$, and using again the Cauchy-Schwarz's inequality for showing that $AB \geq C^2$, it is easy to prove that the previous inequality holds only for odd values of m. Therefore, the fourth-power estimate is the only stable fixed point of the soft iteration.

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